Tiebreaker Round Solutions

AMSA-MAMS Pi Day Mathematics Tournament

March 9, 2019

1. Scrooge McDuck is being held hostage by an angry mathematician. The mathematician writes five distinct numbers a, b, c, d, e on the board. The mathematician knows that by Szekeres's theorem, there are three numbers that, in order, are either all increasing or decreasing (e.g., $a < b < d$ or $b > c > e$, but not $e < a < c$). Scrooge is blindfolded so that he can't see the numbers, but if he can tell the mathematician which numbers satisfy these properties. Luckily, he can bribe the guard *once* for $\$X$. Then, the guard will answer X questions about the order of the numbers (for example, $a > b$ or $b < e$), one by one. What's the smallest amount of money Scrooge must pay in order to guarantee his escape?
   Answer: $\boxed{\$4}$
   Solution: \$3 is not enough. We will show how to guarantee a solution with \$4. First, Scrooge compares (asks the guard to compare) b, c, d (pairwise). If $b < c < d$ or $b > c > d$, he is done. Otherwise, either $c < b < d$ or $d < c < b$. In the first case, Scrooge compares a, b. If $a < b$, he knows that $a < b < d$. If $a > b$, he knows that $a > b > c$. The second case can be solved similarly. Thus, Scrooge needs to ask 4 questions in the worst case.

2. Scrooge McDuck is in trouble with a mathematician again. The mathematician writes five numbers $a < b < c < d < e$ on the board. Scrooge is blindfolded, but he will be freed if he can tell the mathematician whether or not π is written on the board. Luckily, he can bribe the guard *once* for $\$X$. Then, the guard will answer X questions strictly comparing π to a number on the board (for example, $\pi < a$ or $\pi > d$, but not $\pi = b$ or $\pi \leq c$), one by one. What's the smallest amount of money Scrooge must pay in order to guarantee his escape?
   Answer: $\boxed{\$4}$
   Solution:

   - Scrooge compares c, π.
     - If $\pi < c$, the π is either a, b, or none.
     - Scrooge compares π, b.
       * If $\pi < b$, π is either a or none. This takes 2 checks $a < \pi$ and $\pi < a$ to determine in $a = \pi$, for a total of 4.
       * Otherwise, $\pi \geq b$. This π is either b or none. This takes 1 check to determine if $b = \pi$, for a total of 3.
   - Otherwise, $c \leq \pi$
   - Scrooge compares π, d.
     - If $\pi < d$, π is either c or none. This takes 1 more check $c < \pi$ to determine if π is on the board. This takes 3 checks total.
     - Otherwise, $d \leq \pi$
       * If $\pi < e$, Scrooge checks if $\pi = d$ be checking if $d < \pi$. This takes a total of four checks.
       * Otherwise, $e \leq \pi$, so Scrooge checks if $\pi = e$ be checking if $e < \pi$. This takes a total of four checks.

   Across all cases, it takes at most 4 comparisons, so Scrooge will have to pay at most \$4.

3. Scrooge McDuck keeps getting in trouble. The mathematician writes six distinct numbers such that they first increase, then decrease (for example, $a > b > c > d > e > f$ or $a < b < c < d > e > f$. Scrooge is blindfolded, but he will be freed if he can tell the mathematician which of a, b, c, d, e, f is the largest. Luckily, he can bribe the guard *once* for $\$X$. Then, the guard will answer X questions about the order of the numbers (for example, $a > b$ or $b < e$), one by one. What's the smallest amount of money Scrooge must pay in order to guarantee his escape? This process takes at most 3 comparisons.

Answer: $\boxed{\$3}$

Solution: First, Scrooge compares c, d. If $c > d$, then the maximum must be either a, b, or c. If $c < d$, then the maximum must be either d, e, or f. Let x, y, z be the relevant numbers. Then, Scrooge compares y, z. If $y < z$, he is done, and the answer is z (corresponding to either c or f). Otherwise if $y > z$, he must then compare x, y to get the answer.