ω **Omega Round**

AMSA-MAMS Pi Day Mathematics Tournament

March 9, 2019

1. A $9 \times 9 \times 9$ cube is removed from a $10 \times 10 \times 10$ cube. How many $2 \times 2 \times 2$ cubes will fit in the remaining shape?
2. What is the probability that a randomly chosen integer between π and 2π is even?
3. Bob and Joe are spending the hour before the physics test running around in circles instead of studying. It takes Bob 32 minutes to run around in a circle and it takes Joe 48 minutes to run around in a circle. If the both start at the Physics room door together at 5:57, at what time will both meet again at the Physics room door?
4. A boat takes a journey up and down a river in four hours. The boat's average speed was $45mph$, and the speed of the current was $30mph$. How long would the journey take, in hours, in still water?
5. John is wondering how high his fence is. He takes a 1ft piece of cardboard and sees that when it is 2ft from the fence it makes a 3ft shadow. How high is John's fence?
6. How many factors does 4078380 have?
7. Ron and Harry are two wizards with a magic box of super-delicious sweets that have names orange yellow. Every time Ron takes out a candy (he always picks orange candies if he can), a yellow candy is added, and every time Harry takes out a candy (he always picks yellow candies) a orange candy is added to the magic box of super-delicious sweets. The magic box of super-delicious sweets starts off with 20 sweets, 10 orange and 10 yellow. Each minute either Ron takes a sweet or Harry takes a sweet from the magic box, and every three minutes Nor takes takes 2 sweets then Harry takes one. When is the first time Ron takes a yellow candy because there are no more orange candies if they start at 3:15 (the first candy is taken by 3:16)?
8. Let equilateral triangle DEC of area $\sqrt{3}$ be inscribed in rectangle ABCD. Let O be in incenter of the triangle. Find the area of trapezoid AEOD.
9. If the two intersection points of the functions:

   $f(x) = 2x^2 + 3x + 1$

   $f(x) = -5x^2 - 11x + 1$

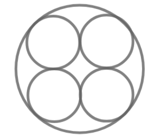
   are (a, b) and (c, d), find $|ad - bc|$, where "$|x|$" indicates the absolute value of x.
10. Kumar Express is travelling at 50 mph. The train has 16 wheels, but hits a bump every 6 minutes and loses half of its wheels. This also causes the train the slow down by 6 mph. The train explodes when there is only one wheel left. How far does the train travel after hitting the first bump?
11. Two thieves stole a long necklace full of big, sparkling diamonds and pearls, and they want to divide the diamonds and pearls between themselves evenly. There are 10 Dazzling Diamonds and 16 Precious Pearls, arranged from left to right in the pattern

    $$PPDPDDDPDPDPPDPPPPDPDDPPPP$$

    The first thief suggested cutting up the necklace into the bits PP, D, P, DDD, P, D, P, D, PP, D, PPPP, D, P, DD, and PPPP using 14 cuts, then splitting up these pieces so that each thief gets 5 Dazzling Diamonds and 8 Precious Pearls. However, the second thief is lazy and does not want to make that many cuts because each cut takes a long time to make. Find the minimum number of cuts you need to cut up the necklace so that each thief gets 5 Dazzling Diamonds and 8 Precious Pearls.

12. Alex is procrastinating as he normally does. He has to finish 10 problems in 30 minutes. However, if he worked by himself, he would finish 7.5 minutes late. But with the help of his brother, he manages to finish all the problems exactly on time. How many problems can Alex finish in the time it takes his brother to finish one problem.

13. Find the radius of the larger circle shown if the radius of the 4 smaller circles internally tangent to it each have a radius of 1. Each smaller circle is tangent to its two adjacent smaller circles.



14. Find the shortest distance from the point (2,2) to the line $20x + 21y + 5 = 0$.

15. Let $f(x^2 + x) = 4x^2 + 4x - 3$. What is $f(5)$?

16. For how many integer values of x between 1 and 99 is the function $f(x) = x^3$ one more than a multiple of 3?

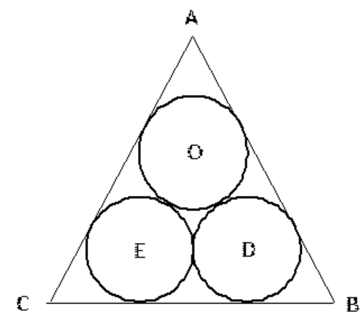
17. If $\frac{1}{x} + \frac{1}{y} = 3$, and $3xy = 2$, compute $\frac{1}{x^2} + \frac{1}{y^2}$.

18. Robert has a total of \$3.14 in pennies, nickels, dimes, and quarters. He has as many pennies as nickels, 2 more quarters than nickels, and as many dimes as pennies, nickels, and quarters combined. How many dimes does Robert have?

19. In a circle with center O, and with 3 points A, B, C in that order which lie on a minor arc of the circle, given that $\angle AOC = 110°$ and $\angle OCB = 45°$, what is the measure of $\angle OAB$?

20. Victor is playing a game with a friend. They have a bag of 20 marbles, 4 of which are red. They take turns drawing marbles. The first person to draw a red marble is the winner. Given that Victor goes first, what is the probability that victor wins on one of his first three turns?

21. Let Circles O, E, and D have area 1 and be tangent to each other and triangle ABC. Find the area of triangle ABC. (See right.



22. Define a function $f(x)$ satisfying $f(f(x)) = f(x + 2)$ having $f(1) = 0$, $f(0) = 4$, and $f(5) = 8$. Find $f(2)$.

23. There is a sequence defined like Fibonacci, where $F_n = F_{n-1} + F_{n-2}$. The seventh term is 20. What is the sum of the first ten terms of the sequence?

24. If $n!$ ends in exactly 100 zeroes, what is the greatest possible value of n?

25. Farmer John wants to build his rectangular fence along a river. If he has 28 yards of fence, what is the maximum area he can enclose in the fence?

26. What is $(1)+(1+2)+(1+2+3)+(1+2+3+4)+(1+2+...+5)+...+(1+2+3+...+218)+(1+2+3+...+218+219)$?

27. There is a square of side length 1. Inside it is inscribed a circle, and inside that another square, and inside that another circle, to infinity. What is the sum of the areas of all the squares?

28. Distinct positive integers a and b have 5 and 6 factors, respectively. What is the smallest possible product ab if a and b are relatively prime?

29. Alex the geologist discovers a shape in the coordinate plane of area 20. He is surprised to note that there are 6 lattice points along its edges. How many lattice points are inside the figure?

30. Find the remainder when $314 + 3^{314}$ is divided by 10.

31. Alice and Bob are playing a game. This time, they only flip one coin. Alice wins if they reach the pattern HT first. Bob wins if they reach the pattern HH first. What's the probability that Alice wins? Note, Alice can win via a sequence like "THT"

32. Alice and Bob are playing another game. They each flip a coin until one of them wins. Alice wins when she flips HT. Bob wins when he flips HH. Let a be the expected number of flips until Alice wins, and let b be the expected number of flips until Bob wins. What is $|a - b|$? Note, Alice can win via a sequence like "THT"

33. What is the sum of the squares of the 7th row of Pascal's triangle, written as a binomial coefficient?

34. What is the number I'm thinking of? If your answer is x and the average answer is y, your score for this problem will be $10 \cdot min\{\frac{x}{y}, \frac{y}{x}\}$

35. How many numbers in the first 50 digits of π are prime?

36. Pick a number n from $\{1, 2, 3, ..., 10\}$. If x teams in total pick the answer n, your team will score $\frac{n}{x}$ points.