ω Omega Round Solutions

AMSA-MAMS Pi Day Mathematics Tournament

March 9, 2019

1. A $9 \times 9 \times 9$ cube is removed from a $10 \times 10 \times 10$ cube. How many $2 \times 2 \times 2$ cubes will fit in the remaining shape?
   $\boxed{0}$

2. What is the probability that a randomly chosen integer between π and 2π is even?
   $\boxed{\frac{2}{3}}$

3. Bob and Joe are spending the hour before the physics test running around in circles instead of studying. It takes Bob 32 minutes to run around in a circle and it takes Joe 48 minutes to run around in a circle. If the both start at the Physics room door together at 5:57, at what time will both meet again at the Physics room door?
   Answer: $\boxed{\textbf{7:33}}$ 96 minutes after 5:57 is 7:33
   Solution: If m minutes pass and m is a multiple of both 32 and 48 minutes, then both Bob and Joe will be at the physics room door. To find the next time they meet at the door we must find the least common multiple of 32 and 48, which is 96. 96 minutes after 5:57 is 7:33.

4. A boat takes a journey up and down a river in four hours. The boat's average speed was $45mph$, and the speed of the current was $30mph$. How long would the journey take, in hours, in still water?
   Answer: $\boxed{\text{3 hours}}$
   Solution: The average speed was 45 mph and the journey was 4 hours, so the total distance traveled was $45 * 4 = 180$ miles, or 90 miles in each direction. Let the speed of the boat be r. Then we have

$$\frac{90}{r+30} + \frac{90}{r-30} = 4$$

, or

$$90(r-30) + 90(r+30) = 4(r+30)(r-30)$$

so

$$180r = 4r^2 - 3600$$

then

$$r^2 - 45r - 900 = 0$$

so

$$(r-60)(r+15) = 0$$

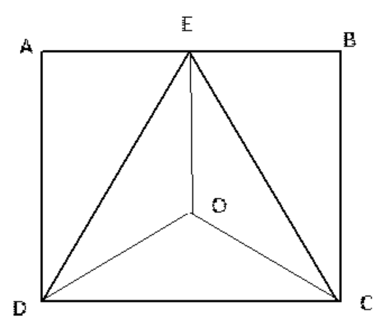
Thus, $r = 60$, so the trip in still water would tale $180/60 = 3$ hours.

5. John is wondering how high his fence is. He takes a 1ft piece of cardboard and sees that when it is 2ft from the fence it makes a 3ft shadow. How high is John's fence?
   Answer: $\boxed{\frac{5}{3}}$
   Solution: By similar triangles, the answer is $1 \cdot \frac{3+2}{3} = \frac{5}{3}$

6. How many factors does 4078380 have?
Answer: $\boxed{48}$
Solution:$4078380 = 2019 * 2020 = 2019 * 202 * 10 = 3 * 673 * 2^2 * 5^1 * 101^1$. Thus, the number has $(1+1)*(1+1)*(2+1)*(1+1)*(1+1) = 48$ factors.

7. Ron and Harry are two wizards with a magic box of super-delicious sweets that have names orange yellow. Every time Ron takes out a candy (he always picks orange candies if he can), a yellow candy is added, and every time Harry takes out a candy (he always picks yellow candies) a orange candy is added to the magic box of super-delicious sweets. The magic box of super-delicious sweets starts off with 20 sweets, 10 orange and 10 yellow. Each minute either Ron takes a sweet or Harry takes a sweet from the magic box, and every three minutes Nor takes takes 2 sweets then Harry takes one. When is the first time Ron takes a yellow candy because there are no more orange candies if they start at 3:15 (the first candy is taken by 3:16)?
Answer: $\boxed{\textbf{3:44}}$
Solution: Every three minutes there is a not loss of 1 orange candy and net gain of 1 yellow candy, so at 3:42 27 minutes have passed, and there is 1 orange candy and 19 yellow candies. At 3:43 Nor takes the last orange candy, so at 3:44 Nor has to take a yellow candy.

8. Let equilateral triangle DEC of area $\sqrt{3}$ be inscribed in rectangle ABCD. Let O be in incenter of the triangle. Find the area of trapezoid AEOD.



Answer: $\boxed{\frac{5\sqrt{3}}{3}}$
Solution: The area of an equilateral triangle is $\frac{s^2\sqrt{3}}{4}$, so solving for s, we get $s = 2$. Then, we have $AE = 1$, $AD = \sqrt{3}$, and $EO = \frac{2\sqrt{3}}{3}$. Then, the area of the trapezoid is $\frac{1}{2} \cdot (\sqrt{(}3) + \frac{2\sqrt{3}}{3}) \cdot 1 = \frac{5\sqrt{3}}{3}$

9. If the two intersection points of the functions:
$f(x) = 2x^2 + 3x + 1$
$f(x) = -5x^2 - 11x + 1$
are (a, b) and (c, d), find $|ad - bc|$, where "$|x|$" indicates the absolute value of x.
Answer: $\boxed{2}$
Solution: $2x^2 + 3x + 1 = -5x^2 - 11x + 1$
$7x^2 + 14x = 0$
$x^2 + 2x = 0$
$x(x + 2) = 0$
$x = -2, 0$
$f(0) = 2 \cdot 0 + 3 \cdot 0 + 1 = 1$ implies $(0, 1)$
$f(-2) = 2 \cdot 4 + 3 \cdot (-2) + 1$ implies $(-2, 3)$

$|ad - bc| = |0 \cdot 3 - 1 \cdot (-2)| = 2$

10. Kumar Express is travelling at 50 mph. The train has 16 wheels, but hits a bump every 6 minutes and loses half of its wheels. This also causes the train the slow down by 6 mph. The train explodes when there is only one wheel left. How far does the train travel after hitting the first bump?
    Answer: $\boxed{\text{11.4 miles}}$
    Solution: After the first bump, the train will have 8 wheels left. The train will go through 3 more bumps, and it travels for 0.1 hours each time. Thus, the total distance traveled will be $0.1 * ((50-6) + (50-12) + (50-18)) = 0.1 * (150 - 36) = 11.4$ miles.

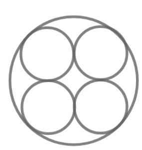
11. Two thieves stole a long necklace full of big, sparkling diamonds and pearls, and they want to divide the diamonds and pearls between themselves evenly. There are 10 Dazzling Diamonds and 16 Precious Pearls, arranged from left to right in the pattern

$$PPDPDDDPDPDPPDPPPPDPDDPPPP$$

    The first thief suggested cutting up the necklace into the bits PP, D, P, DDD, P, D, P, D, PP, D, PPPP, D, P, DD, and PPPP using 14 cuts, then splitting up these pieces so that each thief gets 5 Dazzling Diamonds and 8 Precious Pearls. However, the second thief is lazy and does not want to make that many cuts because each cut takes a long time to make. Find the minimum number of cuts you need to cut up the necklace so that each thief gets 5 Dazzling Diamonds and 8 Precious Pearls.
    Answer: $\boxed{2}$
    Solution: It cannot be done with 1 cut, because then the necklace would have to be cut exactly in the middle so that each person gets 13 jewels, but this would not be an even distribution of diamonds and pearls. It can be done with two cuts, into the the sections PPDPD, DDPDPDPPDPPP, and DPDDPPPP. One thief gets the first and last section while the other gets the middle section.

12. Alex is procrastinating as he normally does. He has to finish 10 problems in 30 minutes. However, if he worked by himself, he would finish 7.5 minutes late. But with the help of his brother, he manages to finish all the problems exactly on time. How many problems can Alex finish in the time it takes his brother to finish one problem.
    Answer: $\boxed{\text{4 problems}}$
    Solution: Alex does 10 problems in 37.5 minutes. Therefore, $\frac{x}{60} = \frac{10}{37.5}$ which means that $x = 16$, which means that Alex can do 16 problems an hour Because they finish exactly on time, we know that Alex's brother does $20 - x$ problems which is 4 problems. Therefore, Alex can finish 4 problems in the time it takes his brother to do one.

13. Find the radius of the larger circle shown if the radius of the 4 smaller circles internally tangent to it each have a radius of 1. Each smaller circle is tangent to its two adjacent smaller circles.



    Answer: $\boxed{1 + \sqrt{2}}$
    Solution: Draw a radius of larger circles to point of tangency. By drawing a square with side length one, you can find out the distance from the center of the larger circle to the center of a smaller one is $\sqrt{2}$. Add 1 to get the radius.

14. Find the shortest distance from the point (2,2) to the line $20x + 21y + 5 = 0$.
$\boxed{3}$
Solution: The distance between a line $Ax + By + C = 0$ and a point (x_0, y_0) is

$$\frac{A(x_0) + B(y_0) + C}{\sqrt{A^2 + B^2}}$$

. Plugging in, we get that the answer is 3.

15. Let $f(x^2 + x) = 4x^2 + 4x - 3$. What is $f(5)$?
$\boxed{17}$
Solution: We notice that $4x^2 + 4x - 3 = 4(x^2 + x) - 3$, so the given function can be written as $f(a) = 4a - 3$, from which we get 17.

16. For how many integer values of x between 1 and 99 is the function $f(x) = x^3$ one more than a multiple of 3?
Answer: $\boxed{33}$
Solution: If x is one more than a multiple of 3, then x^3 is also one more than a multiple of 3. If x is 2 more than a multiple of 3, then x^3 is also 2 more than a multiple of 3. Finally, if x is a multiple of 3, then x^3 is also a multiple of 3. Therefore, all values of x which are 1 more than a multiple of 3 work, which are 1, 4, 7, 10, ... 97. There are 33 numbers in this sequence.

17. If $\frac{1}{x} + \frac{1}{y} = 3$, and $3xy = 2$, compute $\frac{1}{x^2} + \frac{1}{y^2}$.
Answer: $\boxed{6}$
Solution: Squaring $\frac{1}{x} + \frac{1}{y} = 3$ yields $\frac{1}{x^2} + \frac{2}{xy} + \frac{1}{y^2} = 9$, and from $3xy = 2$ we have $\frac{2}{xy} = 3$. Plugging this back into the first equation gives us $\frac{1}{x^2} + \frac{1}{y^2} + 3 = 9$, so $\frac{1}{x^2} + \frac{1}{y^2} = 6$.

18. Robert has a total of \$3.14 in pennies, nickels, dimes, and quarters. He has as many pennies as nickels, 2 more quarters than nickels, and as many dimes as pennies, nickels, and quarters combined. How many dimes does Robert have? **Answer:** $\boxed{14}$ Let the number of pennies Robert has be x. Then he also has x nickels, $x{+}2$ quarters, and $x{+}x{+}x{+}2$ dimes, or $3x{+}2$ dimes. Given pennies are \$0.01, nickels are \$0.05, dimes are \$0.10, and quarters are \$0.25, we have $0.01x + 0.05x + 0.10 * (3x + 2) + 0.25 * (x + 2) = 3.14$.

$$0.01x + 0.05x + 0.30x + 0.20 + 0.25x + 0.50 = 3.14$$

.

$$0.61x + 0.70 = 3.14$$

$$0.61x = 2.44$$

$$x = 4$$

So Robert has 3*(4) + 2 = 14 dimes.

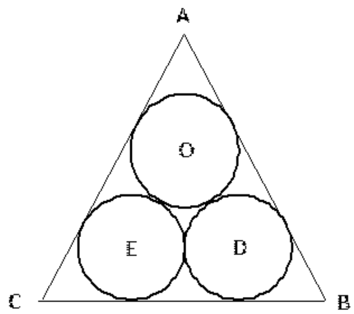
19. In a circle with center O, and with 3 points A, B, C in that order which lie on a minor arc of the circle, given that $\angle AOC = 110°$ and $\angle OCB = 45°$, what is the measure of $\angle OAB$?
Answer: $\boxed{80°}$
Solution: Construct radius OB. Since $\triangle BOC$ is isosceles, $\angle CBO = \angle OCB = 45°$. Therefore, $\angle BOC = 90°$. Since $\angle AOC = 110°$, $\angle AOB = 20°$. Denote x to be $\angle BAO$. Since $\triangle AOB$ is isosceles, $2x + 20 = 180$. Therefore $x = 80°$, which is the answer.

20. Victor is playing a game with a friend. They have a bag of 20 marbles, 4 of which are red. They take turns drawing marbles. The first person to draw a red marble is the winner. Given that Victor goes first, what is the probability that victor wins on one of his first three turns?

Answer: $\boxed{\frac{8}{15}}$

Solution: $\boxed{\frac{8}{15}}$

21. Let Circles O, E, and D have area 1 and be tangent to each other and triangle ABC. Find the area of triangle ABC. **Answer:** $\frac{6+4\sqrt{3}}{\pi}$ **Solution:** Draw line OH perpendicular to AC. Then OH $= \sqrt{\frac{1}{\pi}}$, so



OA $= 2\sqrt{\frac{1}{\pi}}$. Triangle OED is equilateral, so it has height $\sqrt{\frac{3}{\pi}}$. Thus, the total height of the triangle is $\sqrt{\frac{3}{\pi}} + \frac{3}{\sqrt{\pi}}$. Then, the triangle has side length $\frac{2}{\sqrt{\pi}}(1+\sqrt{3})$. Plugging this into the formula for the area of an equilateral triangle, $\frac{s^2\sqrt{3}}{\pi}$, we get the answer $\frac{6+4\sqrt{3}}{\pi}$

22. Define a function $f(x)$ satisfying $f(f(x)) = f(x+2)$ having $f(1) = 0$, $f(0) = 4$, and $f(5) = 8$. Find $f(2)$.
Answer: $\boxed{8}$
Solution: We have $f(f(1)) = f(0) = 4 = f(3)$ and $8 = f(5) = f(3+2) = f(f(3)) = f(4) = f(f(0)) = f(0+2) = f(2)$.

23. There is a sequence defined like Fibonacci, where $F_n = F_{n-1} + F_{n-2}$. The seventh term is 20. What is the sum of the first ten terms of the sequence?
Answer: $\boxed{\textbf{200}}$ Sum of first 10 terms = 220
Solution: Let x be the sixth term and y be the fifth term. Then, the 4th term is $x - y$, the 3rd term is $2y - x$, the 2nd term is $2x - 3y$, and the 1st term is $5y - 3x$. Also, the 8th term is $20 + x$, the 9th is $40 + x$, and the 10th is $60 + 2x$. The sum of all the numbers is then $x - y + 2y - x + 2x - 3y + 5y - 3x + 20 + 20 + x + 40 + x + 60 + 2x$, which simplifies to $3y + 3x + 140$. We know by the recursive formula that $x + y = 20$, so the total is $3 * 20 + 140 = 200$

24. If $n!$ ends in exactly 100 zeroes, what is the greatest possible value of n? **Answer:** $\boxed{409}$

25. Farmer John wants to build his rectangular fence along a river. If he has 28 yards of fence, what is the maximum area he can enclose in the fence?
Answer: $\boxed{\text{98 yards}}$.
Solution: One neat trick to this problem is to allocate the same amount of fence to each dimension. Thus, the length will get 14 yards combined, and the width will get 14 yards *combined.* Thus, the dimension will be $7x14$, so the maximal area will be 98 yards.

26. What is $(1) + (1+2) + (1+2+3) + (1+2+3+4) + (1+2+...+5) + ... + (1+2+3+...+218) + (1+2+3+...+218+219)$?
Answer: $\boxed{1774630}$ **Solution:** The sum can be written as $219 \cdot 1 + 218 \cdot 2 + ... + 1 \cdot 219 = 220 \cdot 1 - 1 \cdot 1 + 220 \cdot 2 - 2 \cdot 2 ... + 220 \cdot 219 - 219 \cdot 219 = 220 \cdot (1 + 2 + ... + 219) - (1^2 + 2^2 + ... + 219^2) = (220)\frac{219 \cdot 220}{2} - \frac{219 \cdot 220 \cdot (2*219+1)}{6} = 1774630$

27. There is a square of side length 1. Inside it is inscribed a circle, and inside that another square, and inside that another circle, to infinity. What is the sum of the areas of all the squares?
Answer: $\boxed{\textbf{2}}$ Area of squares $= 2$
Solution: In general, for a square of side length x, the circle inside has a radius of $\frac{x}{2}$. The square inscribed inside of that circle would have a diagonal of length x, so its side length would be $\frac{x}{\sqrt{2}}$. That means that its area would be $\frac{x^2}{2}$, exactly one half of the area of the original square. That means we simply need to add $1 + \frac{1}{2} + \frac{1}{4} \ldots$, which by the geometric series formula is 2.

28. Distinct positive integers a and b have 5 and 6 factors, respectively. What is the smallest possible product ab if a and b are relatively prime?
Answer: $\boxed{ab = 720}$
Solution: We must have $a = p^4$ and either $b = q^5$ or $b = q^1r^2$. To get the smallest product, we set $p = 2$, and let $b = 5 \cdot 3^2$. Then, $ab = 720$.

29. Alex the geologist discovers a shape in the coordinate plane of area 20. He is surprised to note that there are 6 lattice points along its edges. How many lattice points are inside the figure?
Answer: $\boxed{18}$ Number of points inside $= 18$
Solution: By Pick's theorem, $A = i + b/2 - 1$. We know A = 20 and b = 6, so i = 18.

30. Find the remainder when $314 + 3^{314}$ is divided by 10. **Answer:** $\boxed{3}$
Solution: $3^4 = 1$ mod 10, so $3^{314} = 3^(78 * 4 + 2) = 3^2 = 9$. Then, the answer is $314 + 9 = 3$ (mod 10), so the remainder is 3.

31. Alice and Bob are playing a game. This time, they only flip one coin. Alice wins if they reach the pattern HT first. Bob wins if they reach the pattern HH first. What's the probability that Alice wins? Note, Alice can win via a sequence like "HHT."
Answer: $\boxed{0.5}$
Solution: Let p_* be the expected number of flips for Alice to win, given the sequence so far is $*$. Then,

$$p = 0.5p_H + 0.5p_T = 0.5p_H + 0.5p$$

$$p_H = 0.5p_{HH} + 0.5p_{HT} = 0.5(0) + 0.5(1) = 0.5$$

Solving, we get $p = p_H = 0.5$.

32. Alice and Bob are playing another game. They each flip a coin until one of them wins. Alice wins when she flips HT. Bob wins when he flips HH. Let a be the expected number of flips until Alice wins, and let b be the expected number of flips until Bob wins. What is $|a - b|$? Note, Alice can win via a sequence like "HHT."
Answer: $\boxed{2}$
Solution: Let a_* be the expected number of flips for Alice to win, given the sequence so far is $*$. Then,

$$a = 0.5(a_H + 1) + (a_T + 1) = 0.5(a_H + 1) + 0.5(a + 1)$$

$$a_H = 0.5(a_{HT} + 1) + 0.5(a_{HH} + 1) = 0.5(a_{HT} + 1) + 0.5(a_H + 1)$$

$$a_{HT} = 0$$

Solving, we get $a_H = 2$ and $a = 4$.

Similarly, for Bob, we have

$$b = 0.5(b_H + 1) + 0.5(b_T + 1) = 0.5(b_H + 1) + 0.5(b + 1)$$

$$b + H = 0.5(b_{HH} + 1) + 0.5(b_{TH} + 1) = 0.5(b_{HH} + 1) + 0.5(b + 1)$$

$$b_{HH} = 0$$

Solving, we get $b = b_H + 2$ and $b_H = 0.5b + 1$, so $b = 6$. Thus, the answer is 2.

33. What is the sum of the squares of the 7th row of Pascal's triangle, written as a binomial coefficient?
Answer: $\boxed{14C7}$ $Sum of squares = 14C7$
Solution: The sum of squares of any row n of Pascal's triangle is $2nCn$.

34. What is the number I'm thinking of? If your answer is x and the average answer is y, your score for this problem will be $10 \cdot min\{\frac{x}{y}, \frac{y}{x}\}$

35. How many numbers in the first 50 digits of π are prime? **Answer:** $\boxed{23}$

36. Pick a number n from $\{1, 2, 3, ..., 16\}$. If x teams in total pick the answer n, your team will score $\frac{n}{x}$ points.