β Beta Round Solutions

AMSA-MAMS Pi Day Mathematics Tournament

March 9, 2019

1. John is 2 years older than twice George's age. 4 years ago, John was 2 years younger than four times George's age. How old will John be in four years?
   Answer: $\boxed{22}$
   Solution: Let George's age be g. Then we have $j = 2 + 2g$ and $j - 4 = 4(g - 4) - 2 = 4g - 18$. Thus, $2g - 2 = 4g - 18$, so $2g = 16$, so $g = 8$. Thus, John is $2 * 8 + 2 = 18$ right now, so in four years, he will be 22.

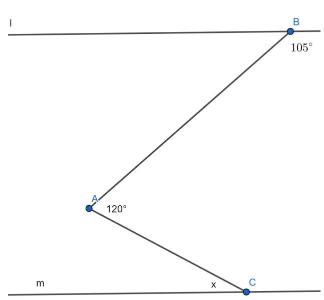
2. Find base b such that such that $1_b + 2_b + 3_b + 10_b = 56_{101}$
   Answer: $\boxed{505}$
   Solution: In base 10, the equation is $6 + (1 * b + 0) = 5 * 101 + 6$, so $b = 5 * 101 = 505$.

3. See Figure (a). If $l \parallel m$, find x.
   Answer: $\boxed{x = 45}$
   Solution: Extend AB to meet line m at D. By parallel lines, angle D is congruent to the supplement of angle B. Thus, $< ADC = 180 - 105 = 75°$. We also have $< DAC = 180 - 120 = 60°$. Since the angles in a triangle add to $180°$, we must have $x = 45$.

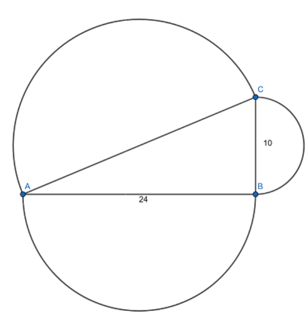
4. See Figure (b). Given $< ABC$ is a right angle, find the sum of the areas of the three semicircles.
   Answer: $\boxed{169\pi}$
   Solution: By the Pythagorean theorem, we have $Ac = 26$. Then, the total area is $0.5(13^\pi + 5^2\pi + 12^2\pi) = 0.5 * (2 * 13^2\pi) = 169\pi$.

5. The second term of a geometric sequence is 20. The eighth term is 1280. What is the eleventh term?
   Answer: $\boxed{10240}$
   Solution: If the ratio between the terms is r, the ratio between the eighth and second term will be $r^6 = (1280/20) = 64$, so $.r = \pm 2$. Then, the eleventh term will be ± 10240.

6. What are the last two digits of 2018^{20}
   Answer: $\boxed{76}$ Last two digits = 76
   Solution: We can find a pattern by looking at the last two digits of the first few powers of 18, which will have the same two last digits as the powers of 2018.
   The cycle is (18), 24, 32, 76, 68, 24..., so we can see that the cycle starts at index 2 and ends at index



(a)



(b)

5. In general, the cycle starts at index 2 mod 4 and ends at 1 mod 4. That means that 2018^18 will start the cycle with 24, so 2018^{20} ends in 76.

7. Neptune's castle by special octopuses with 7, 8, 9, and 10 arms. Octopuses with an even number of arms always tell the truth and octopuses with an odd number of arms always lie. One day a red, green, yellow, blue, and purple octopus are gathered together. The red octopuses says "I have 8 arms, and the green octopus has 9 arms." The yellow octopus says "Everyone but me is lying." The green octopus says "The red octopus is correct and the purple and blue octopuses are lying." The purple octopus says "No one has 9 or 10 arms here." The blue octopus chooses to remain silent because they do not want to distinguish themselves. What is the sum of the number of arms of the octopuses? **Answer:** $\boxed{\mathbf{37}}$

   Solution: If the red octopus is telling the truth, then the green octopus would be lying, so the red octopus would be wrong. Thus, the red octopus is lying and so is the green octopus. Since the green octopus cannot have 9 arms, they have 7 arms. Also, the purple and blue octopuses must be telling the truth. Then the purple and blue octopuses must tell the truth, so they have 8 arms, and everyone else is lying and has 7 arms. The sum of the number of arms is 8+8+7+7+7=37.

8. There is a weighted six-sided die such that the probability of rolling each n from 1 through 6 inclusive is proportional to n^2. What is the probability of rolling the same number twice?

   Answer: $\boxed{\frac{25}{91}}$

   Solution: $1^2+2^2+3^2+4^2+5^2+6^2 = 6(13)(7)/6 = 91$, so the probability of rolling any number is $\frac{n^2}{91}$. That means that the probability of rolling doubles is $\sum_{n=1}^{6} \frac{n^4}{91^2}$, which is simply $\frac{1+16+81+256+625+1296}{8281} = \frac{25}{91}$

9. Andy is walking in a field of flowers. He notices that if he arranges the flowers in groups of 11, he will have 1 too many flowers. He also realizes that if he arranges the flowers in groups of 15, he will have 1 too few flowers. What is the smallest number of flowers that can be in Andy's field?

   Answer: $\boxed{89}$

   Solution: We know that the number of flowers is 1 less than a number divisible by 15, so we can say that the number of flowers must end in a 9 or 4. We also know that the number of flowers must be 1 greater than a multiple of 11. Therefore, we know that we must multiply 11 by a number ending in 3 or 8 and add 1 to get the number of flowers. Through trial and error, we only need to check two solutions 34 (which does not work) and 89 (which does work). Therefore, Andy's field has 89 flowers in it.

10. Factor the expression $x^6 - x^4 + 64x^2 - 64$

    Answer: $\boxed{(x+1)(x-1)(x^2+4x+8)(x^2-4x+8)}$

    Solution: First, we rewrite the expression as $x^4(x^2-1) + 64 * (x^2-1) = (x^4+64)(x^2-1)$. We can factor x^2-1 into $(x+1)(x-1)$ using difference of squares. Surprisingly, x^4+64 can be factored more. We can apply the Sophie-Germain identity: $a^4 + 4b^4 = (a^2+2b^2+2ab)(a^2+2b^2-2ab)$. Thus, $x^4 + 4 * 2^4 = (x^2 + 2 * 2^2 + 2 * x * 2)(x^2 + 2 * 2^2 - 2 * x * 2)$. Thus, $x^6 - x^4 + 64x^2 - 64 = (x+1)(x-1)(x^2+4x+8)(x^2-4x+8)$