# $\alpha$ Alpha Round Solutions

AMSA-MAMS Pi Day Mathematics Tournament

March 9, 2019

1. Simplify $\sqrt{3^3 + 6(3^2)}$
   **Answer:** $\boxed{9}$
   **Solution:** $\sqrt{3^3 + 6(3^2)} = \sqrt{27 + 54} = \sqrt{81} = 9$

2. It takes 7 Pokeman raisers 3 minutes to raise a Pokeman's stage 4 times. How many Pokeman raisers are needed to raise a Pokeman's stage 314 times within 1 hour?
   **Answer:** $\boxed{28}$
   **Solution:** Within 1 hour = 60 minutes, 7 Pokeman raisers can raise a Pokeman by $4 * 20 = 80$ stages. We need 314 stages, so we need at least $7 * (314/80)$ Pokeman raisers. We can multiply and divide out, but also note that the answer must be a whole number. $314/80 \approx 4$, so the answer is 28.

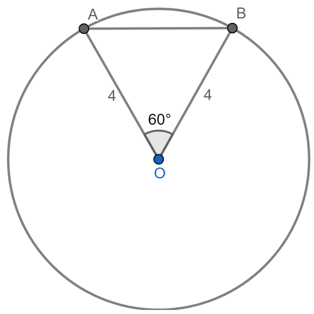
3. Five students take a test. Four of the scores are 75, 85, 95, and 72. If the average score is 80, what is the score of the last student?
   **Answer:** $\boxed{73}$
   **Solution:** Let the score be $x$. Then, the average of the five students is $80 = (75+85+95+72+x)/5 = (327 + x)/5$. Multiplying both sides by 5 we get $400 = 327 + x$, so $x = 73$.

4. Thomas has a 5 mL mixture of 50% water and 50% soda. How much water must he add to get a mixture of 25% soda?
   **Answer:** $\boxed{5mL}$
   **Solution:** Thomas has $0.5 * 5 = 2.5$ mL of soda. This is 25% of the final mixture, so the final mixture must have 10mL of liquid. Thus, Thomas must add 5mL of water to the mixture.

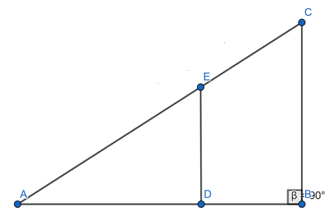
5. In a circular arc $AOB$ of measure $60°$, with $AO = BO = 4$, what is the area of the sector between $\overset{\frown}{AB}$ and $AB$? (see Figure (a))
   **Answer:** $\boxed{\frac{8\pi}{3} - 4\sqrt{3}}$
   **Solution:** The area of the sector $AOB$ is $\frac{1}{6}$ of the area of the circle, so it is $\frac{16\pi}{6} = \frac{8\pi}{3}$. We now have to subtract the area of $\triangle AOB$. $AOB$ is equilateral since $AO = BO$ and $\angle O = 60°$, so using the area of an equilateral triangle formula, we find that $\triangle AOB$ has area $4\sqrt{3}$. Therefore, the desired area is $\frac{8\pi}{3} - 4\sqrt{3}$.

6. In Figure (b), $AC = 8, AD = 3, ED = 4$, and $ED \| CB$. Find $BC$
   **Answer:** $\boxed{\frac{32}{5}}$
   We can get the answer using similar triangles.

7. Given Max has 4 random pieces of pie, each either apple, pecan, or, blueberry, what is the probability that Max can enjoy at least 2 slices of different flavor?
   **Answer:** $\boxed{\frac{26}{27}}$
   **Solution:** We count the opposite: the probability that all of his slices are the same flavor. They can all be the same flavor if they are all apple (probability of $(\frac{1}{3})^4$), pecan (probability of $(\frac{1}{3})^4$), and blueberry (probability of $(\frac{1}{3})^4$). Adding the probabilities, we obtain $\frac{1}{27}$. But since this counts the opposite of the desired probability, we subtract this probability from 1 to obtain $\frac{26}{27}$.




(a)




(b)

8. The number $10^{10} - n$, for some integer $n$, is divisible by 7. What is the minimum value of $n$?
**Answer:** $\boxed{4}$
**Solution:** Consider $10^{10}$ mod 7. $10 \equiv 3$, so $10^{10} \equiv 3^{10} \equiv 9^5 \equiv 2^5 \equiv 32 \equiv 4 \pmod 7$. Thus, $10^{10} - 4 \equiv 0 \pmod 7$, so the answer is 4.

9. There are 2 chairs in the front row of a theater. Each row of chairs has two times one more than the number of chairs of the row in front of it. If there are 10 rows in the theater, how many chairs are there in the theater in total?
**Answer:** $\boxed{\mathbf{4072}}$ Number of chairs = 4072
**Solution:** Explicit formula: $2^{n+1} - 2$ chairs in each row
$S = 2^2 - 2 + 2^3 - 2 + 2^4 - 2 \ldots + 2^{11} - 2$
$S + 20 = 2^2 + 2^3 + 2^4 \ldots + 2^{11}$
$2S + 40 = 2^3 + 2^4 \ldots + 2^{12}$
$S + 20 = 2^{12} - 2^2$
$S + 20 = 4096 - 4$
$S = 4072$

10. Jack and Jill are playing a competitive game. Each round, each player decides to say either 1 or 2 simultaneously. Let the sum of the numbers that round be $a$. If $a$ is even, Jack gets $a$ points and Jill gets $-a$ points. But if $a$ is odd, Jack gets $-a$ points and Jill gets $a$ points. If Jack and Jill play for 100 rounds, what is the expected value of Jack's score (assuming both players play optimally).
**Answer:** $\boxed{\frac{-25}{3}}$
**Solution:** To play optimally, both players must have random strategies. Suppose that Jack chooses 2 with probability $p$ and that Jill chooses 2 with probability $q$. Then, Jack's expected score for one round will be:
$$4pq - 3(1-p)(q) - 3(1-q)(p) + 2(1-p)(1-q)$$
, or
$$12pq - 5p - 5q + 2$$
Jacks want to maximize this quantity while Jill wants to minimize this quantity Rewriting $12pq - 5p - 5q + 2$, we get
$$p(12q - 5) - 5q + 2$$
We have three options
   (a) $12q - 5 = 0$, so $q = \frac{5}{12}$, and Jack's expected score is $\frac{-1}{12}$.
   (b) $12q - 5 < 0$. Then, Jack will set $p = 0$ (otherwise he will lose points on average). Then, Jack's expected score will be $2 - 5q$. However, since $12q - 5 < 0$, the value of $q$ will be smaller than the case above, so Jack will get more points than in the case above. Thus, the first case is strictly better for Jill.
   (c) $12q - 5 > 0$. Then, Jack will set $p = 1$ (he will gain points on average). Then, Jack will get $7q - 3$ points on average. Right now, the score to beat is $\frac{-1}{12}$. If $7q - 3 < \frac{-1}{12}$, we must have $q < \frac{5}{12}$. Thus, there is no value of $q$ where $12q - 5 > 0$ and Jack's expected score is less then $\frac{-1}{12}$.

Then, after 100 rounds, Jack's expected score will be $100 * \frac{-1}{12} = \frac{-25}{3}$.