ω Omega Round

AMSA-MAMS Pi Day Mathematics Tournament

March 10, 2018

1. Jack wants to make noodles, but he only has X hours before he has to serve dinner to his friends Bob and Marley. He decides to recruit the help of his younger brother Frank to make the noodles. Jack can prepare 6 bowls of noodles in 2 hours, and Frank can prepare 1 bowl of noodles in 1 hour. If Bob and Marley will eat 8 bowls of noodles each, what is the minimum value of X that Jack and Frank need?

2. Define $A \star B = A \cdot B + 1$ and $A \triangle B = A^2 \cdot B$. Find $(4 \triangle 2) \star (3 \triangle 7)$

3. Using two 5's and two 6's it is possible to create four positive 4-digit numbers that are divisible by 11. What is the sum of these four numbers?

4. Call a "switch" on the string *abcde* when you switch two adjacent elements. For example turning *abcde* into *bacde* is a switch. What is the minimum number of switches it takes to turn $MAMSA$ into $AMSAM$?

5. Aditya reads one page on Day 1, two pages on Day 2, four pages on Day 3, 8 pages on day 4 and so on. On what day does he reach 1023 total pages read?

6. Tom Brady gets 30,000 yards the first game of his career. However, unfortunately, every game after that he gets half the yards he got the game before. If he plays forever (which, in real life, he probably will), how many yards will he accumlate in his career?

7. Danush takes 5 basketball shots. If there is a 32/243 chance he misses all 5, what is the probability he makes a shot? Express your answer as a common fraction.

8. What is the sum of the squares of the roots of $f(x) = x^2 - x - 99$?

9. Both Shane and John each roll a special 7 sided die which has the numbers 1, 2, 3, π, 4, 5, and 6 on its faces. Assuming there is an equal probability of rolling each number, what is the probability that the sum of both dice rolls is not an even number? Write your answer as a simplified fraction.

10. Let right triangle ABC be inside its circumcircle P. If $m\angle A = 30°$, $m\angle B = 90°$ and $AB = 7$, find the radius of P.

11. Several couples arrive at a party. Each person shakes the hand of every other person, excluding their spouse. If there were a total of 112 handshakes, how many couples attended the party?

12. Consider the regular hexagon $PIEDAY$. What fraction of the area of $PIEDAY$ is occupied by triangle PAD?

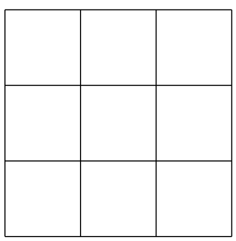
13. Let ΔTOM be a triangle with $TO = 8$, $TM = 15$, and $OM = 17$. Let J be the center of the circle circumscribed about ΔTOM. Let E be the point on OM such that $TE \perp OM$. Find JE. Your answer should be in the form $\frac{m}{n}$ where m and n are relatively prime integers.

14. Find the difference between the sum of the first 100 multiples of 811 and the first 99 multiples of 809. Assume 811 and 809 are both the first multiple of themselves.

15. Aditya has 3 green beads and 7 red beads. How many ways can he make a distinct bracelet? (2 bracelets are the same if one can be rotated to obtain the other)

16. Aditya is particularly fond of Tom Brady. Because of this, he finds the value of 12!. He then finds the sum of the digits of that value. He then finds the sum of the digits of that value. He continues this process until he is left with a single digit number. What number is he left with?

17. A square is divided into 9 squares as shown. How many rectangles can be formed by connecting any number of adjacent square(s)?



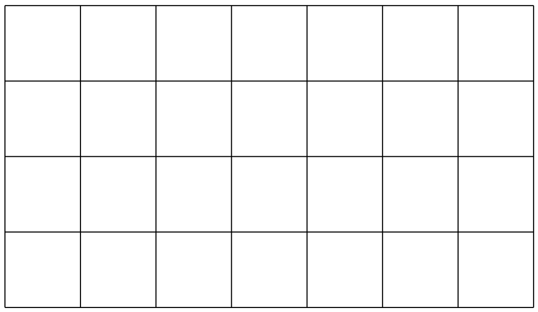
18. Let $f(x)$ be a quadratic function such that $f(1) = 3$, $f(2) = 5$, and $f(3) = 11$. Find the constant term of $f(x)$.

19. Brian writes down all of the natural numbers, skipping every number that has 12 factors. Charles writes down all of the natural numbers, skipping every number ending in 3. What is the smallest natural number that neither Brian nor Charles write?

20. Steven and Olaf take turns writing numbers on the board. Steven begins by writing 1011_2. Then Olaf treats it as a base-3 number and adds 2, before writing it down in base 3. Then Steven treats the base-3 number Olaf writes as a base 4 number and adds 3, before writing his result in base 4. If they continue this process, let S be the first number Steven writes down whose value is greater than 100_{10}, and let L be the first number Olaf writes down whose value is greater than 100_{10}. Find S+L in base 10.

21. Tom and Brady take turns rolling 2 standard 6-sided dice. The winner is the first person to roll a sum of 12. Let m be the probability Tom wins if he rolls first and let n be the probability Tom wins if he rolls second. What is $m - n$?

22. Marie wants to walk home. However, in order to get there, she must navigate a 7 by 4 grid as below. She begins in the bottom left square and may only move upwards or to the right along the gridlines. How many different paths can she take to get to the top right square?



23. For a point P in the plane, let $d_1(P)$ and $d_2(P)$ be the distances of the point P from lines $x - y = 0$ and $x + y = 0$, respectively. Calculate the area of the region R consisting of all points P lying in the first quadrant of the plane and satisfying $2 \leq d_1(P) + d_2(P) \leq 4$.

24. Distinct positive integers a and b have 5 and 6 factors, respectively. What is the smallest possible product ab if a and b are relatively prime?

25. Kenny rolls 101 fair 6-sided dice. What is the probability that an even number of the dice have an even number on the side facing up?

26. Given that 4003997 is the product of two prime factors, find the largest prime factor of 4003997.

27. Let $f(x)$ be a quadratic function such that $f(-1) = 3$, $f(1) = 9$ and $f(2) = 15$. Find the sum of the squares of roots of $f(x)$.

28. $n!$ ends in exactly 100 zeroes, what is the greatest possible value of n?

29. An unbiased coin is tossed. If the result is a head, a pair of an unbiased dice is rolled and sum of numbers on the two faces is noted. If the results is a tail, a card from well shuffled deck pack of eleven cards numbered $2, 3, 4, ..., 12$ is picked and the number on the card is noted. What is the probability that the noted number is a 7 or an 8?

30. If

$$S_n = \sum_{k=2}^{4n} (-1)^{\frac{k\cdot(k+1)}{2}} \cdot k^2$$

then find S_n

31. Find all x that satisfy the equation

$$16^x + 3 \cdot 2^{2x} + 1 = 2^{3x+2} + 2^{x+2}$$

32. Let (a, b, c) be the ordered triple chosen from $20^{18}, 18^{20}, 201^8$ such that $a < b < c$. Compute the last digit of $314a + 159b + 265c$.

33. Tarang is at the origin. Every second, he moves one unit to the right with probability 1/2 and to the left with probability 1/2. He wins if he reaches (2,0) and loses if he reaches (-1,0). What is the probability that he wins?

34. Julia notices that she has a large number of eggs, and she tries to divide them into a number of baskets. She tries dividing them into 2 baskets but has 1 egg left over. She tries dividing them into 3 baskets but has 2 eggs left over. She tries dividing them into 5 baskets but has 3 eggs left over. She tries dividing them into 7 baskets but has 4 eggs left over. She tries dividing them into 11 baskets but has 5 eggs left over. At this point, she gives up. What is the least amount of eggs she could have?

35. If $a \in \mathbb{R}$ and the equation $-3(x - [x])^2 + 2(x - [x]) + a^2 = 0$ has no integral solution, then calculate all possible values of a. Note: $[x]$ denotes the greatest integer $\leq x$.

36. If the integers m and n are chosen at random between 1 and 100 inclusive, then calculate the probability that a number of the form $7^m + 7^n$ is divisible by 5 .