ω Omega Round Solutions

AMSA-MAMS Pi Day Mathematics Tournament

March 10, 2018

1. Jack wants to make noodles, but he only has X hours before he has to serve dinner to his friends Bob and Marley. He decides to recruit the help of his younger brother Frank to make the noodles. Jack can prepare 6 bowls of noodles in 2 hours, and Frank can prepare 1 bowl of noodles in 1 hour. If Bob and Marley will eat 8 bowls of noodles each, what is the minimum value of X that Jack and Frank need?

   Answer: $\boxed{4}$ Jack makes 3 bowls an hour, and Frank makes 1 bowl an hour, so together, they make 4 bowls an hour. They need to make $8 \times 2 = 16$ bowls, so they need $16 \div 4 = 4$ hours to do it.

2. Define $A \star B = A \cdot B + 1$ and $A \triangle B = A^2 \cdot B$. Find $(4 \triangle 2) \star (3 \triangle 7)$

   Answer: $\boxed{2017}$

$$
\begin{aligned}
&(4 \triangle 2) \star (3 \triangle 7) \qquad (1)\\
&= (4^2 \cdot 2) \star (3^2 \cdot 7)\\
&= 32 \star 36\\
&= 32 \cdot 63 + 1\\
&= 2017
\end{aligned}
$$

3. Using two 5's and two 6's it is possible to create four positive 4-digit numbers that are divisible by 11. What is the sum of these four numbers?

   Answer: $\boxed{24442}$ 5566+6655+6556+5665=24442

4. Call a "switch" on the string *abcde* when you switch two adjacent elements. For example turning *abcde* into *bacde* is a switch. What is the minimum number of switches it takes to turn $MAMSA$ into $AMSAM$?

   Answer: $\boxed{3}$ $MAMSA \to AMMSA \to AMSMA \to AMSAM$

5. Aditya reads one page on Day 1, two pages on Day 2, four pages on Day 3, 8 pages on day 4 and so on. On what day does he reach 1023 total pages read?

   Answer: $\boxed{10}$ After n days, he has read $2^n - 1$ pages.

6. Tom Brady gets 30,000 yards the first game of his career. However, unfortunately, every game after that he gets half the yards he got the game before. If he plays forever (which, in real life, he probably will), how many yards will he accumlate in his career?

   Answer: $\boxed{60000}$ The sum of a geometric sequence with first term 30,000 and common ratio $\frac{1}{2}$ is 60000.

7. Danush takes 5 basketball shots. If there is a 32/243 chance he misses all 5, what is the probability he makes a shot? Express your answer as a common fraction.

   Answer: $\boxed{\frac{1}{3}}$ Since Danush takes 5 shots, if the probability of missing a shot is x then the probability that he misses all 5 shots is x^5 which is given as $\frac{32}{243}$. Therefore, the probability that he misses a single shot is $x = \frac{2}{3}$ so the probability that he makes a shot is $1 - x = \frac{1}{3}$.

8. What is the sum of the squares of the roots of $f(x) = x^2 - x - 99$?

   Answer: $\boxed{199}$ Let the roots be r_1 and r_2. Using Vieta's formulas, $r_1 + r_2 = 1$ and $r_1 \times r_2 = -99$. So, ${r_1}^2 + {r_2}^2 = (r_1 + r_2)^2 - 2r_1r_2 = 1^2 - 2(-99) = 199$.

9. Both Shane and John each roll a special 7 sided die which has the numbers 1, 2, 3, π, 4, 5, and 6 on its faces. Assuming there is an equal probability of rolling each number, what is the probability that the sum of both dice rolls is not an even number? Write your answer as a simplified fraction.

   Answer: $\boxed{\frac{31}{49}}$ This is the same thing as the probability that the sum is an even number subtracted from 1. The two numbers have to both be odd or even for the sum to be even, so there are $3 \cdot 3 + 3 \cdot 3 = 18$ outcomes, since there are 3 available odd numbers and even numbers. There are a total of $7 \cdot 7 = 49$ possible outcomes, so the desired outcomes is $49 - 18 = 31$. The probability would then be $\frac{31}{49}$

10. Let right triangle ABC be inside its circumcircle P. If $m\angle A = 30°$, $m\angle B = 90°$ and $AB = 7$, find the radius of P.

    Answer: $\boxed{\dfrac{7\sqrt{3}}{3}}$ Note that the radius is equal to the length of the other leg of the triangle.

11. Several couples arrive at a party. Each person shakes the hand of every other person, excluding their spouse. If there were a total of 112 handshakes, how many couples attended the party?

    Answer: $\boxed{\text{8 couples}}$ Let there be n people. Each person shakes hands with $n - 2$ people resulting in $n(n - 2)$ handshakes. However, this is double counted meaning that it should be divided by 2. Therefore, there are $\frac{n(n-2)}{2}$ handshakes total. Since there are 112 total handshakes, $\frac{n(n-2)}{2} = 116$. Therefore, n=16 meaning that there are 8 couples.

12. Consider the regular hexagon $PIEDAY$. What fraction of the area of $PIEDAY$ is occupied by triangle PAD?

    Answer: $\boxed{\dfrac{1}{3}}$ Let the side length of the hexagon be s. Then, the area of the hexagon is $\frac{3s^2\sqrt{3}}{2}$. Consider triangle PAD. Angle A is a right angle, and PA is $s\sqrt{3}$. Therefore, the area of triangle PAD is $s \cdot s\sqrt{3} \cdot \frac{1}{2}$ meaning the triangle has area $\frac{s^2\sqrt{3}}{2}$. From here, it is clear that triangle PAD occupies $\frac{1}{3}$ of the area of the hexagon

13. Let ΔTOM be a triangle with $TO = 8$, $TM = 15$, and $OM = 17$. Let J be the center of the circle circumscribed about ΔTOM. Let E be the point on OM such that $TE \perp OM$. Find JE. Your answer should be in the form $\frac{m}{n}$ where m and n are relatively prime integers.

    Answer: $\boxed{\frac{161}{34}}$ First, notice that ΔTOM is a right triangle with right angle at T. So, OM is a diameter of the circumscribed circle, so JM and JT are both radii of the circle. Thus, $JT = JM = \frac{17}{2}$. $ET \times OM = TM \times OT = 2 \times Area_{\Delta TOM}$. Thus, $ET = \frac{120}{17}$. ΔETJ is a right triangle with right angle at E. $JT = \frac{17}{2}$, $ET = \frac{120}{17}$, so $JE = \frac{161}{34}$.

14. Find the difference between the sum of the first 100 multiples of 811 and the first 99 multiples of 809. Assume 811 and 809 are both the first multiple of themselves.

    Answer: $\boxed{91000}$ Sum of the first 100 multiples of 811 can be written as

    $811(1 + 2 + 3... + 98 + 99) + 81100$

    Sum of the first 99 multiples of 809 can be written as

    $809(1 + 2 + 3... + 98 + 99)$

    Subtract the two

    $811(1 + 2 + 3... + 98 + 99) + 81100 - 809(1 + 2 + 3... + 98 + 99)$

    $=(811 - 809)(1 + 2 + 3... + 98 + 99) + 81100$

    $=2(1 + 2 + 3... + 98 + 99) + 81100$

$=2[(1+99)+(2+98)+(3+97)...+(48+52)+(49+51)+50]+81100$

$=2(100\cdot 49+50)+81100$

$=2(4950)+81100=91000$

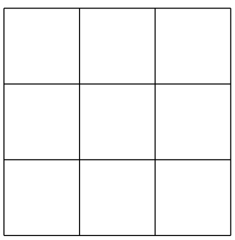
15. Aditya has 3 green beads and 7 red beads. How many ways can he make a distinct bracelet? (2 bracelets are the same if one can be rotated to obtain the other)

    Answer: $\boxed{12}$ $\frac{_{10}C_3}{10}=12$

16. Aditya is particularly fond of Tom Brady. Because of this, he finds the value of 12!. He then finds the sum of the digits of that value. He then finds the sum of the digits of that value. He continues this process until he is left with a single digit number. What number is he left with?

    Answer: $\boxed{9}$ 12! is divisible by 9, so each time he takes the sum of the digits it will remain divisible by 9, and the final number will be 9.

17. A square is divided into 9 squares as shown. How many rectangles can be formed by connecting any number of adjacent square(s)?

    Answer: $\boxed{36}$ To choose a rectangle is to choose the borders of the rectangle. Have to pick 2 out of 4 horizontal borders and 2 out of 4 vertical borders. There are 6 ways to choose 2 borders for each, so the total combinations is $6\cdot 6=36$

18. Let $f(x)$ be a quadratic function such that $f(1)=3$, $f(2)=5$, and $f(3)=11$. Find the constant term of $f(x)$.

    Answer: $\boxed{5}$ Let $f(x)=ax^2+bx+c$. Then, $a+b+c=3$, $4a+2b+c=5$, and $9a+3b+c=11$. Solving these equations, we obtain a constant term, or $c=5$.

19. Brian writes down all of the natural numbers, skipping every number that has 12 factors. Charles writes down all of the natural numbers, skipping every number ending in 3. What is the smallest natural number that neither Brian nor Charles write?

    Answer: $\boxed{1323}$ If Brian skips all numbers that have 12 factors, then those numbers can be written either as a^3*b^2, a^5*b^1, or a^{11}, where a and b are primes. Since we want to find the smallest of these numbers ending in 3, we should focus on the numbers in the form of a^3*b^2. We know that 2 cannot be a prime factor, because the number would be even and would not end in 3. We also know 5 cannot be a prime factor, because the number would end in 5. So we look at the next two smallest primes, 3 and 7. We want to let 3 have the bigger power, to minimize the number. So if we let a=3, and b=7, we get $3^3*7^2=1323$, which does end in 3 and therefore is the smallest number.

20. Steven and Olaf take turns writing numbers on the board. Steven begins by writing 1011_2. Then Olaf treats it as a base-3 number and adds 2, before writing it down in base 3. Then Steven treats the base-3 number Olaf writes as a base 4 number and adds 3, before writing his result in base 4. If they continue this process, let S be the first number Steven writes down whose value is greater than 100_{10}, and let L be the first number Olaf writes down whose value is greater than 100_{10}. Find S+L in base 10.

    Answer: $\boxed{383}$.

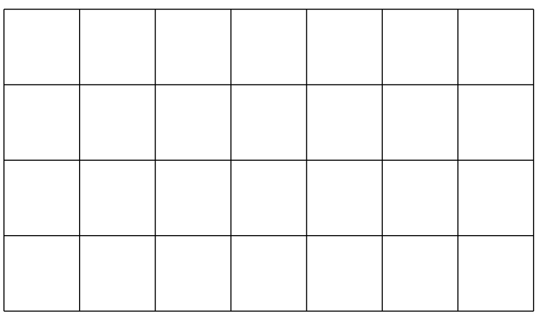
Steven	1011_2	23
Olaf	1020_3	33
Steven	1023_4	75
Olaf	1032_5	142
Steven	1041_6	241

We see that $L = 142$ and $S = 241$. Thus $S + L = 241 + 142 = 383$.

21. Tom and Brady take turns rolling 2 standard 6-sided dice. The winner is the first person to roll a sum of 12. Let m be the probability Tom wins if he rolls first and let n be the probability Tom wins if he rolls second. What is $m - n$? Express your answer as a common fraction.

    Answer: $\boxed{\frac{1}{71}}$ The probability Tom wins on the first turn is $\frac{1}{36}$. The probability Tom doesn't win on the first turn and proceeds to win on his second turn is $\frac{35}{36} \times \frac{35}{36} \times \frac{1}{36}$. By continuing this pattern, it is clear that m is the sum of an infinite geometric sequence with first term $\frac{1}{36}$ and common ratio $(\frac{35}{36})^2$, which is $\frac{36}{71}$. $n = 1 - m$, so $n = \frac{35}{71}$. Therefore, $m - n = \frac{1}{71}$.

22. Marie wants to walk home. However, in order to get there, she must navigate a 7 by 4 grid as below. She begins in the bottom left square and may only move upwards or to the right along the gridlines. How many different paths can she take to get to the top right square?



    Answer: $\boxed{84}$ Let the bottom left corner be $(0, 0)$ and the number of ways to get to a square (x, y) be $s(x, y)$. Then observe that $s(x, y) = s(x - 1, y) + s(x, y - 1)$. If we rotate the grid, we see that it is identical to Pascal's Triangle, and the entry we are looking for $\binom{9}{3} = 84$.

23. For a point P in the plane, let $d_1(P)$ and $d_2(P)$ be the distances of the point P from lines $x - y = 0$ and $x + y = 0$, respectively. Calculate the area of the region R consisting of all points P lying in the first quadrant of the plane and satisfying $2 \leq d_1(P) + d_2(P) \leq 4$.

    Answer: $\boxed{6}$ Distance between a point and a line is $\frac{Ax+By+C}{\sqrt{A^2+B^2}}$. Plugging this into the restraint, we get $2 \leq \frac{|x-y|}{\sqrt{2}} \frac{|x+y|}{\sqrt{2}} \leq 4$ If $x >= y$, we get $x \in [\sqrt{2}, 2\sqrt{2}]$. If $x <= y$, $y \in [\sqrt{2}, 2\sqrt{2}]$. These two regions form a square with its lower left corner missing. This, the area of rhe region is $(2 * \sqrt{2})^2 - (\sqrt{2})^2 = 6$

24. Distinct positive integers a and b have 5 and 6 factors, respectively. What is the smallest possible product ab if a and b are relatively prime?

    Answer: $\boxed{720}$ $a = 2^4$, $b = 5 \cdot 3^2$

25. Kenny rolls 101 fair 6-sided dice. What is the probability that an even number of the dice have an even number on the side facing up?

    Answer: $\boxed{\dfrac{1}{2}}$ By symmetry, the probability of obtaining a evens, where a is even, is the same as obtaining a odds. Thus, the probability is $\dfrac{1}{2}$.

26. Given that 4003997 is the product of two prime factors, find the largest prime factor of 4003997.

    Answer: $\boxed{2003}$ Observe that $4003997 = (2001 - 2) \cdot (2001 + 2)$. Thus, 2001+2 $= 2003$.

27. Let $f(x)$ be a quadratic function such that $f(-1) = 3$, $f(1) = 9$ and $f(2) = 15$. Find the sum of the squares of roots of $f(x)$.

Answer: $\boxed{-1}$ Solving the system of equations, we find that $f(x) = x^2+3x+5$. The sum of the roots is $r_1+r_2 = -3$ and the product of the roots is $r_1r_2 = 5$. Thus, $r_1^2+r_2^2 = (r_1+r_2)^2-2r_1r_2 = 9-10 = -1$.

28. $n!$ ends in exactly 100 zeroes, what is the greatest possible value of n?

Answer: $\boxed{409}$ If $n!$ ends in exactly 100 zeroes, there must be 100 factors of 5 in the numbers 1 to n. It can be seen that the number 405 satisfies this condition as $\frac{405}{5} + \lfloor \frac{405}{25} \rfloor + \lfloor \frac{405}{125} \rfloor = 100$. From 406 to 409, no new factors of 5 are added. Therefore, the largest value is 409.

29. An unbiased coin is tossed. If the result is a head, a pair of an unbiased dice is rolled and sum of numbers on the two faces is noted. If the results is a tail, a card from well shuffled deck pack of eleven cards numbered $2, 3, 4, ..., 12$ is picked and the number on the card is noted. What is the probability that noted number is a 7 or an 8? Express your answer as a common fraction.

Answer: $\boxed{\frac{193}{792}}$ The probability that the coin toss results in heads is $\frac{1}{2}$ and the probability that the die roll a 7 or an 8 is $\frac{11}{36}$, resulting in a probability of $\frac{11}{72}$. The probability that the coin flips tails is $\frac{1}{2}$ and the probability that the card picked is a 7 or an 8 is $\frac{2}{11}$ resulting in a total probability of $\frac{1}{11}$. The sum of these probabilities is $\frac{11}{72} + \frac{1}{11} = \frac{193}{792}$

30. If $S_n = \sum_{k=2}^{4n} (-1)^{\frac{k\cdot(k+1)}{2}} \cdot k^2$ then find S_{32}.

Answer: $\boxed{1056}$ $S_n = -(1)^2 - 2^2 + 3^2 + 4^2 - 5^2 - 6^2 + 7^2 + 8^2 ... = (3^2 - 1^2) + (4^2 - 2^2) + (7^2 - 5^2) + (8^2 - 4^2) ... = 2((4+12+...)_{nterms} + (6+14+...)_{nterms}) = 2(n(4+4n-4)+n(6+4n-4) = 4n(4n+1)$. For $n = 32, 1056 = 32 * 33$

31. Find all x that satisfy the equation

$$16^x + 3 \cdot 2^{2x} + 1 = 2^{3x+2} + 2^{x+2}$$

Answer: $\boxed{0}$ Simplify and we have $(2^x - 1)^4 = 0$. So, $x = 0$.

32. Let (a, b, c) be the ordered triple chosen from $20^{18}, 18^{20}, 201^8$ such that $a < b < c$. Compute the last digit of $3141a + 5926b + 5358c$.

Answer: $\boxed{9}$ The ordered triple (a, b, c) is $(201^8, 20^{18}, 18^{20})$. The last digits of these numbers make up the ordered triple $(1, 0, 6)$. Therefore, the last digit of the answer is 1+0+8=9

33. Tarang is at the origin. Every second, he moves one unit to the right with probability 1/2 and to the left with probability 1/2. He wins if he reaches (2,0) and loses if he reaches (-1,0). What is the probability that he wins?

Answer: $\boxed{\frac{1}{3}}$ We compute the probability that he loses and use complementary counting to find the probability that he wins. He loses on the first turn with probability 1/2. He can lose on the third turn with probability 1/8, on the 5th turn with probability 1/32, and so on. Thus the probability that he loses is $\frac{1}{2} + \frac{1}{8} + \frac{1}{32} + \cdots = \frac{2}{3}$, so he wins with probability $\frac{1}{3}$

34. Julia notices that she has a large number of eggs, and she tries to divide them into a number of baskets. She tries dividing them into 2 baskets but has 1 egg left over. She tries dividing them into 3 baskets but has 2 eggs left over. She tries dividing them into 5 baskets but has 3 eggs left over. She tries dividing them into 7 baskets but has 4 eggs left over. She tries dividing them into 11 baskets but has 5 eggs left over. At this point, she gives up. What is the least amount of eggs she could have?

Answer: $\boxed{1523}$ We can use Chinese Remainder Theorem to work our way up. A number that is 1 mod 2 and 2 mod 3 must be 5 mod 6. A number that is 5 mod 6 and 3 mod 5 must be 23 mod 30. A number that is 23 mod 30 and 4 mod 7 must be 53 mod 210. A number that is 53 mod 210 and 5 mod 11 must be 1523 mod 2310. Thus, the smallest possible amount of eggs is 1523.

35. If $a \in \mathbb{R}$ and the equation $-3(x-[x])^2+2(x-[x])+a^2=0$ has no integral solution, then calculate all possible values of a. Note: $[x]$ denotes the greatest integer $\leq x$.

Answer: $\boxed{(-1,0)\cup(0,1)}$ Let $t = x - \lfloor x$. Then, $-3t^2+2t+a^2=0$, so $t = \frac{1\pm\sqrt{1+3a^2}}{3}0$. Since $0 \leq t < 1$, we get $\sqrt{1+3a^2} < 2$, or $1+3a^2 < 4$. Then, $a^2 < 1$, or $(a+1)(a-1) < 0$. Thus, $a \in (-1,0)\cup(0,1)$.

36. If the integers m and n are chosen at random between 1 and 100 inclusive, then calculate the probability that a number of the form $7^m + 7^n$ is divisible by 5.

Answer: $\boxed{\frac{1}{4}}$ 7^a mod 5 is either $2, -1, -2, 1$ no matter the value of m, there are exactly 25 values for n. Thus the anser is $\frac{25}{100}$, or $\frac{1}{4}$