Tiebreaker Solutions

AMSA-MAMS Pi Day Mathematics Tournament

March 11, 2017

1. Let x_1, x_2, x_3 be the roots of $x^3 - 6x^2 + 25x + 7$. Compute $2(x_1 + x_2)(x_2 + x_3)(x_3 + x_1)$.

   Answer: $\boxed{314}$ Let $p(x) = x^3 - 4x^2 + 13x + 7$ Note that the expression is equal to $2((x_1 + x_2 + x_3) - x_3)(((x_1 + x_2 + x_3)) - x_1)((x_1 + x_2 + x_3) - x_2)$. By Vieta's Formulas, this is equal to $2(6 - x_3)(6 - x_2)(6 - x_1) = 2p(6) = 314$

2. Andrew Wiles is trying to climb up 12 stairs. He can take steps of either 1, 2, or 3 stairs at a time. How many different ways can he climb up the steps? One possible way is $3, 3, 3, 3$ Note: The path $2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1$ is different than the path $1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 2$.

   Answer: $\boxed{927}$ Lets call F_n the number of ways he can climb n stairs. $F_1 = 1$, $F_2 = 2$, and $F_3 = 4$. To climb 4 steps, he can either climb 1 then handle the last 3, climb 2, then handle the last 2, or climb 3 then handle the last 1. So, $F_4 = F_3 + F_2 + F_1$. To generalize, $F_n = F_{n-1} + F_{n-2} + F_{n-3}$. Doing this out for $n = 12$ gives $F_n = 927$.

3. Let $\triangle PIE$ be a right triangle such that $PI = 21$ and $IE = 20$ and there is a right angle at I. Let P_1 be the foot of the altitude from I to PE. Construct the altitude from P_1 to IE such that the foot of the altitude is I_1. Continue this process infinitely. If m/n is the sum of the lengths $IP_1, P_1I_1, \ldots$ such that m and n are integers and $\gcd(m, n) = 1$, find the sum of m and n.

   Answer: $\boxed{143}$. Notice that we can construct a geometric series whose first term is $\frac{20 \cdot 21}{29}$ and ratio is $\frac{29-20}{29}$, and its sum is $\frac{140}{3}$

4. Danush wants to write his name on a test, but he is unable to spell. How many ways can he misspell his new name so that none of the letters appear in the correct position?

   Answer: $\boxed{265}$ We need to find the number of derangements of 6 elements, so using PIE we have $!6 = 6! - 6 * 5! + 15 \cdot 4! - 20 \cdot 3! + 15 \cdot 2! - 6 \cdot 1! + 1 = 720 - 720 + 360 - 120 + 30 - 6 + 1 = 265$.