# Ω Omega Round Solutions

AMSA-MAMS Pi Day Mathematics Tournament

March 11, 2017

1. [5] Praneet eats $\frac{1}{4}$ of a pie in 12 seconds, and Danush eats $\frac{1}{7}$ of a pie in 7 seconds. Who eats faster?

   **Answer:** $\boxed{\text{Praneet}}$ Praneet eats $\frac{1}{48}$ pies per second and Danush eats $\frac{1}{49}$ pies per second.

2. [5] Pie is sold in boxes of 6, 12 and 24 slices. What is the minimum number of boxes needed to buy to get 90 slices of pie? **Answer:** $\boxed{5}$ $90 = 3(24) + 1(12) + 1(6) \Rightarrow 3 + 1 + 1 = 5$

3. [5] A sphere with diameter $x$ has a volume of $\pi$. Find the volume of a sphere with radius $x$. **Answer:** $\boxed{8\pi}$

4. [6] Maryam Mirzakhani wants to leave the store. However, in order to do so, she needs to choose the correct door from three choices. Door A reads "I am the correct door." Door B reads "Door C is the correct door." Door C reads "A is the correct door." Exactly one of the doors is telling the truth. Which door should she take?

   **Answer:** $\boxed{C}$ If A is correct, then both A and C are telling the truth. If B is correct, then all of the doors are lying. If C is correct, then only B is telling the truth, which is what we want.

5. [6] Rishi and Praneet went to a rock concert together. Praneet bought a T-shirt, 2 posters, and a pair of sunglasses at the merch table for \$53. Rishi bought 4 posters and 3 pairs of sunglasses. If sunglasses cost twice as much as posters, and T-shirts cost \$28 more than posters, how much money did Rishi and Praneet spend together?

   **Answer:** $\boxed{103}$ Let $t$ = cost of a t-shirt, $p$ = cost of a poster, and $s$ = cost of sunglasses. Then:

   $$t + 2p + s = 63$$

   $$t = p + 28$$

   $$s = 2p$$

   Substituting the second and third equations into the first one:

   $$(p + 28) + 2p + (2p) = 53$$

   $$p = 5$$

   $$s = 10$$

   $$t = 33$$

   Rishi spent $4(5) + 3(10) = \$50$, so their total was $\$50 + \$53 = \$103$

6. [6] Find the sum of the factors of 60.

   **Answer:** $\boxed{168}$ $(1 + 2 + 4)(1 + 3)(1 + 5) = 7 \cdot 4 \cdot 6 = 168.$

7. [7] The mean of a set of 6 numbers $\{18, 50, 62, x-10, x, 2x\}$ is 38. Find the value of $x$.

   **Answer:** $\boxed{27}$ $\frac{50+62+2x+18+x-10+x}{6} = 38$ so $x = 27$

8. [7] Aditya wants to climb the Leaning Tower of Pisa, which is 200 meters high. Every 45 minutes, he climbs 10 meters, but he needs to rest for 15 minutes, falling 5 meters. How many hours does it take him to climb the tower? Do not round your answer.

   **Answer:** $\boxed{\dfrac{155}{4}}$ He gets to the 190 meter mark after 38 hours, and then it takes him another 45 minutes to reach the top, so we have $38 + \frac{3}{4} = \frac{155}{4}$.

9. [7] What is the smallest positive integer that is divisible by the first 10 natural numbers?

   **Answer:** $\boxed{2520}$ $1 \times 2^3 \times 3^2 \times 5 \times 7 = 2520$

10. [8] Charan was supposed to to write 15 problems for the Pi Day Mathematics Tournament by Sunday. Unfortunately he was only done with 2 problems by Sunday. On Monday, he wanted to make up for his incompetence, so he started writing problems. At the end of the day, he said, "I have written at least a third of the total number of problems everyone, including myself, has written." If there were 87 total problems done by Sunday, including his 2, and no one else wrote problems on Monday, what is the minimum number of problems Charan wrote on Monday?

    **Answer:** $\boxed{41}$

11. [8] Srinu needs to tie 300 shoelaces. He can tie the shoelaces in 5 hours if he works alone. If Sathwik can tie 150 shoelaces in 5 hours, how many hours will it take them to tie 300 shoelaces if they work together?

    **Answer:** $\boxed{\dfrac{10}{3}}$ They combine to complete $\frac{1}{5} + \frac{1}{10} = \frac{3}{10}$ of the task per hour, so it takes $\frac{10}{3}$ hours to complete the task.

12. [8] The decimal expansion of 1/7 repeats with period 6. The decimal expansion of 1/7 is 0.142857... Find the period of the repeating part of the decimal expansion of 1/13.

    **Answer:** $\boxed{6}$ We know that the length of the repeating part is the order of 10 mod 13, which is 6. Alternatively, we can divide 1 by 13 and get the same result.

13. [9] A coin of radius 1 cm is tossed onto a surface completely tiled with perfect squares with side length 10 cm. What is the probability that the coin lands completely within one of the tiles?

**Answer:** $\boxed{\dfrac{16}{25}}$

14. [9] Patrick plays video games for 5 minutes on April 1. Every day, he plays 5 more minutes than the day before. For how many minutes does he play video games during the month of April?

**Answer:** $\boxed{2325}$ The total number of minutes is $30\frac{5+150}{2} = 15 \cdot 155 = 2325$.

15. [9] Let a *not nice* chess board be an 8x8 chessboard with one pair of opposite corners removed. How many ways are there to tile a *not nice* chessboard with 31 standard 2x1 dominoes?

**Answer:** $\boxed{0}$ Taking two opposite corners off leaves 32 squares of one color and 30 of the other. However, when you lay down a domino, it covers two squares of opposite colors. Thus, after laying down 30 dominoes, the 2 remaining squares are of the same color. Therefore, the two remaining squares cannot be covered by a domino.

16. [10] Innia needs to write a 5-syllable line for a haiku. Words can be either 1 , 2, or 3 syllables. All words with the same number of syllables are identical. In how many ways can she write her line if the order of the words matters?

**Answer:** $\boxed{13}$ The partitions of 5 into 1, 2, and 3 are $3+2$, $3+1+1$, $2+2+1$, $2+1+1+1$, and $1+1+1+1+1$, so we have $2+3+3+4+1 = 13$ ways for Innia to write her poem.

17. [10] Fermat has 16 feet of fence and wants to make a rectangular pen for his coyotes. He will use the barn as one side of the pen, so the fence only needs to be used for the remaining three sides. What are the dimensions of the pen with the largest area that he can make?

**Answer:** $\boxed{32}$ Let the fence have two sides of length $a$ and one side of length $b$. Then $2a+b = 16$, so by AM-GM, $\frac{2a+b}{2} = 8 \geq \sqrt{2ab}$, so $64 \geq 2ab$, and thus $32 \geq ab$, which is the area of the rectangle. Equality holds when $2a = b$, so the maximum area is indeed 32.

18. [10] Solve $2^{2^{2^x}} = 16$ where $x$ is a real number.

**Answer:** $\boxed{1}$ $2^{2^{2^1}} = 16$.

19. [11] Cedric Villani wants to watch an hour-long episode of anime while doing his homework. Every hour, he watches one-half of the remaining portion of the episode. How many hours of anime does he watch in 8 hours?

**Answer:** $\boxed{\dfrac{255}{256}}$ In 8 hours he watches $\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^7} + \frac{1}{2^8} = \frac{2^8-1}{2^8} = \frac{255}{256}$ hours of anime.

20. [11] Let right triangle $ABC$ be inside its circumcircle $P$. If $m\angle A = 30°$, $m\angle B = 90°$ and $AB = 7$, find the radius of $P$.

**Answer:** $\boxed{\dfrac{7\sqrt{3}}{3}}$ Note that the radius is equal to the length of the other leg of the triangle.

21. [11] Marie wants to walk home. However, in order to get there, she must navigate a 7 by 4 grid. She begins in the bottom left square and may only move upwards or to the right by one square each step. How many paths can she take to get to the top right square?

**Answer:** $\boxed{84}$ Let the bottom left corner be $(0, 0)$ and the number of ways to get to a square $(x, y)$ be $s(x, y)$. Then observe that $s(x, y) = s(x-1, y) + s(x, y-1)$. If we rotate the grid, we see that it is identical to Pascal's Triangle, and the entry we are looking for $\binom{9}{3} = 84$.

22. [12] Andrew Wiles is at the origin of the coordinate plane. Every second, he moves one unit to the right with probability 1/2 and to the left with probability 1/2. He wins if he reaches (2,0) and loses if he reaches (-1,0). What is the probability that he wins?

    **Answer:** $\boxed{\frac{1}{3}}$ We compute the probability that he loses and use complementary counting to find the probability that he wins. He loses on the first turn with probability 1/2. He can lose on the third turn with probability 1/8, on the 5th turn with probability 1/32, and so on. Thus the probability that he loses is $\frac{1}{2} + \frac{1}{8} + \frac{1}{32} + \cdots = \frac{2}{3}$, so he wins with probability $\frac{1}{3}$

23. [12] Find the largest natural number $n$ relatively prime to 8 that divides 2016. Note: if two numbers $a$,$b$ are relatively prime, then $\gcd(a, b) = 1$.

    **Answer:** $\boxed{63}$. If $n$ is coprime to 8 it is coprime to any power of 2. We know that $2016 = 32 \cdot 63$. Since $n|2016$ and $n$ is coprime to 32, then $n|63$. Thus $n \leq 63$, so the maximum value of $n$ is 63.

24. [12] What is the smallest integer with 12 positive factors?

    **Answer:** $\boxed{60}$ A number with prime factorization $p \times q \times r^2$, where $p$, $q$, and $r$ are 3 distinct primes, has 12 factors. The smallest such number is $5 \times 3 \times 2^2 = 60$.

25. [13] Terence Tao is opening an ice cream store with 12 flavors to choose from. He is only allowing customers to get 3 or 4 scoops of ice cream. If customers are allowed to repeat flavors, and the order in which the scoops are on the cone doesn't matter, how many possible combinations are there?

**Answer:** $\boxed{1729}$ $\binom{14}{3}+\binom{15}{4}=1729$ (Look up "stars and bars").

26. [13] A perfectly circular pie is cut in a way that one square and one triangle are circumscribed by the crust. Let $k$ be the largest value of the minimum possible distance between the vertices in degrees. Find the value of $k$.

**Answer:** $\boxed{15^\circ}$ In general, if $\gcd(m,n)=1$, then the minimum such value of $k$ for the $m$-gon and and $n$-gon would be $\frac{180}{mn}$ degrees (See if you can prove it!).

27. [13] $PPPIEEEE$ is the string formed by using the letter $P$ 3 times, the letter $I$ 1 time, and the letter $E$ 4 times. Compute the number of different arrangements of this string.

**Answer:** $\boxed{280}$ Try looking for the "Mississippi Formula:" $\frac{8!}{3!1!4!}=\frac{8\cdot 7\cdot 6\cdot 5}{6}=280.$

28. [14] Let $ABCD$ be a square of side-length 1. Let $E$ be the interior point of square $ABCD$ such that $DCE$ is an equilateral triangle. Find the measure of angle $\angle BAE$ in degrees.

**Answer:** $\boxed{15^\circ}$ Note that $m\angle ADE=30^\circ$ and that $AD=DE$. Because $\triangle ADE$ is isosceles, $m\angle DAE=90-15=75^\circ$. So, $m\angle BAE$ is the complementary angle, $m\angle BAE=15^\circ$.

29. [14] Find the value of $x$ if

$$x=\sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\cdots}}}$$

**Answer:** $\boxed{2}$ Note that the exponent of the first $\sqrt{2}$ can be rewritten as $x$. So, we have $x=(\sqrt{2})^x$. Since there is only one solution, $x=2$ is the only solution.

30. [14] If $x+\frac{1}{x}=5$, then find the value of $x^3+\frac{1}{x^3}$.

**Answer:** $\boxed{110}$ Note that $\left(x+\frac{1}{x}\right)^3=x^3+3\cdot\left(x+\frac{1}{x}\right)+\frac{1}{x^3}=x^3+\frac{1}{x^3}+15=125\Rightarrow x^3+\frac{1}{x^3}=110.$

31. [15] In triangle $AMC$, let $\angle M$ be a right angle and $AM=20$, $MC=21$. Let $P$ be on side $AC$ such that $m\angle CMP=45^\circ$. Find $CP-AP$.

**Answer:** $\boxed{\frac{29}{41}}$ By the Pythagorean Theorem, $AC=29$. Then, by the angle bisector theorem, we know that $\frac{AP}{CP}=\frac{AM}{CM}=\frac{20}{21}$. Then we find that $CP-AP=\frac{21\cdot 29}{41}-\frac{20\cdot 29}{41}=\frac{29}{41}$.

32. [15] Let

$$x=6+\cfrac{1}{3+\cfrac{1}{6+\cfrac{1}{3+\cfrac{1}{6+\cdots}}}}$$

Find $(x-3)^2$

**Answer:** $\boxed{11}$ Note that

$$x=6+\cfrac{1}{3+\cfrac{1}{x}}$$

Then, we move solve for $x$ and find that $x = 3 + \sqrt{11}$. Thus $(x-3)^2 = (\sqrt{11})^2 = 11$.

33. [15] Let $P$ be a circle such that point $P$ is the center of the circle. Let $E$ be a point outside the circle such that a line tangent to circle $P$ intersects the circle at point A. Construct a secant line from $E$ such that it intersects circle $P$ at two distinct points $S$ and $T$ such that $S$ is between $E$ and $T$. If the length of $EA$ is 6 units and the length $ST$ is 5 units, find the length of $ES$.

    **Answer:** $\boxed{\text{4 units}}$ Note that Power of a Point tells us that $6^2 = ET \cdot ES$. Note that if $x = ES$, then $x + 5 = ET$. So, we have: $x(x+5) = 36 \Rightarrow x^2 + 5x - 36 = 0 \Rightarrow x = -9, x = 4$. This tells us that the positive answer is 4.

34. [20] Paul Erdos has two bags of marbles. Bag 1 contains 4 red marbles and 7 blue marbles. Bag 2 contains 5 red marbles and 3 blue marbles. Erdos chooses a bag at random and randomly selects 2 marbles without replacement. What is the probability that he chose Bag 2 if he chooses 1 red marble and 1 blue marble?

    **Answer:** $\boxed{\frac{825}{1609}}$ Let $A$ represent the event that Glenn chooses Bag 2 and let $B$ represent the event that Glenn chooses 1 of each type of marble. The question is equivalent to asking for $P(A|B)$. By Bayes' Theorem, this quantity is equal to $\frac{P(B|A)P(A)}{P(B)}$. $P(A)$ is simply $\frac{1}{2}$. The probability that he chooses 1 of each color from Bag 2 is $\frac{5 \cdot 3}{\binom{8}{2}} = \frac{15}{28}$. Now, we must calculate $P(B)$. Note that $P(B) = \frac{1}{2} \cdot \frac{15}{28} + \frac{1}{2} \cdot \frac{28}{55}$.

    Thus, we have:

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)} = \frac{\frac{15}{28} \cdot \frac{1}{2}}{\frac{1}{2} \cdot \frac{15}{28} + \frac{1}{2} \cdot \frac{28}{55}} = \frac{825}{1609}$$

35. [20] Marie goes to a bakery. The bar sells pies that costs \$2, \$3, \$4, and \$5. She has a total of \$29. In how many ways can she buy pies, where the order in which the pies are bought does not matter?

    **Answer:** $\boxed{63}$. Note that we partition 29 into the set $\{2, 3, 4, 5\}$, and the generating function is

$$\frac{1}{1-x^2} \frac{1}{1-x^3} \frac{1}{1-x^4} \frac{1}{1-x^5}$$

    which is equal to

$$\left(\sum_{i=0}^{\infty} x^{2i}\right) \left(\sum_{i=0}^{\infty} x^{3i}\right) \left(\sum_{i=0}^{\infty} x^{4i}\right) \left(\sum_{i=0}^{\infty} x^{5i}\right)$$

    so using the geometric series identity, truncating, and expanding, we get that the coefficient of $x^{29}$ is 63. Alternatively, casework also works and takes a long time.

36. [20] Estimate the value of the following expression:

$$(\ln(2017))^{\pi}$$

    Let $x$ be your answer to this question and $y$ be the correct value. You will receive $\max(0, \lfloor 20 - 4 \ln |y - x| \rfloor)$ points on this question.

    **Answer:** $\boxed{587.27023026660995557\ldots}$