α Alpha Round Solutions

AMSA-MAMS Pi Day Mathematics Tournament

March 11, 2017

1. [2] Chewing gum is sold in packs of 7, 21 and 28 pieces. Innia buys a total of 91 pieces. What is the smallest number of packs she could have bought?

   Answer: $\boxed{4}$ Maximum number of 28 packs possible is 3, giving 84 pieces of gum. 91-84=7, allowing only 1 7 pack of gum. 1 pack + 3 packs= 4 packs of gum.

2. [3] Define $x \otimes y = x^2 - y^2 + 2$. Find $(4 \otimes 3) \otimes 3$.

   Answer: $\boxed{74}$. We have $(4^2 - 3^2 + 2) \otimes 3 = 9 \otimes 3 = 9^2 - 3^2 + 2 = 81 - 9 + 2 = 74$.

3. [4] Find the sum of the mean, median, and mode of $1, 3, 8, 8, 10$.

   Answer: $\boxed{22}$ Mean: 6, Median: 8, Mode: 8.

4. [4] Suppose that there are 3 distinct digits represented by the symbols A, M, S, all greater than 1. We are given two equations:

$$\begin{array}{ccccc} & \text{A} & \text{M} & \text{S} & \text{A} \\ + & \text{M} & \text{A} & \text{M} & \text{S} \\ \hline 1 & \text{M} & \text{M} & 0 & \text{M} \end{array}$$

$$A = M + S$$

   What is the value of M?

   Answer: $\boxed{4}$

5. [5] The sum of 7 consecutive odd integers is 903. What is the sum of the 6 smallest numbers?

   Answer: $\boxed{768}$ Let middle integer be x. then:

$$(x-6) + (x-4) + (x-2) + x + (x+2) + (x+4) + (x+6) = 903$$

$$7x = 903$$

$$x = 129$$

   Therefore, the largest number $(x + 6)$ will be 129+6=135. 903-135=768

6. [5] Choose $a_1, a_2, \ldots, a_6$ without replacement from the set $\{3, 4, 6, 7, 9, 10\}$ and $b_1, b_2, \ldots, b_6$ without replacement from the set $\{1, 2, 5, 8, 11, 12\}$. Find the maximum value of

$$\sum_{i=1}^{6} a_i b_i$$

   Answer: $\boxed{316}$ By the rearrangement inequality, the maximum value is $10 \cdot 12 + 9 \cdot 11 + 7 \cdot 8 + 6 \cdot 5 + 4 \cdot 2 + 3 \cdot 1 = 120 + 99 + 56 + 30 + 8 + 3 = 316$.

7. [6] How many consecutive zeroes does 1212! end in? Note: ! denotes factorial. For example, $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$.

Answer: $\boxed{300}$ The number of zeroes at the end of a number n is the exponent of the largest power of 5 that evenly divides n. The largest such exponent for 1212 is 300.

8. [6] A club has three boys and six girls. In how many ways can a group of five club members be chosen if there must be at least one person of each gender?

Answer: $\boxed{120}$ We use complementary counting. There are $\binom{9}{5}$ ways to choose 5 people from 9, but we cannot choose all girls, so there are $\binom{9}{5} - \binom{6}{5} = 126 - 6 = 120$ ways to choose the members.

9. [7] Riemann has n pies that he wants to sell to his friends. If he fills his boxes with 7 pies, there are 6 pies left over. If he fills his boxes with 8 pies, there are 2 pies left over. If he fills his boxes of 9 pies, there are 8 pies left over. Find the smallest possible value of n.

Answer: $\boxed{314}$ Using the Chinese Remainder Theorem, one can find that there is only one answer (mod 504), namely 314.

10. [8] Let $PIEDAYMT$ be a rectangular prism such that $PIED$ and $YMTA$ are rectangular faces that are parallel to each other so that each vertex of the rectangle $PIED$ is connected with the corresponding vertex on $YMTA$ (P corresponds to Y and so on) by an edge. Let $PD = 3$, $DA = 1$, and $AT = 4$. Also, let the midpoint of AT be K. If PT and IK intersect at point X, compute the length of XK.

Answer: $\boxed{\frac{\sqrt{14}}{3}}$ Consider the trapezoid $PITK$. Note that triangles $\triangle PIX$ and $\triangle TKX$ are similar so that the sides of $\triangle PIX$ are twice the lengths of the corresponding sides of $\triangle TKX$. So, XK is equal to $\frac{1}{2}$ of the length of IX. In other words, XK is one-third of the length fo IK. So, we compute the value of IK :

$$IK = \sqrt{IM^2 + MK^2} = \sqrt{IM^2 + TK^2 + YA^2} = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}.$$

This means that the length of XK is $\frac{\sqrt{14}}{3}$.